# Unsolvable Problems
Part Two

Outline for Today

- ***More on Undecidability***

  - Even more problems we can't solve.

- ***A Different Perspective on RE***

  - What exactly does "recognizability" mean?

- ***Verifiers***

  - A new approach to problem-solving.

- ***Beyond RE***

  - A beautiful example of an impossible problem.

Recap from Last Time

```
bool willAccept(string function, string input) {
    // Returns true if function(input) returns true.
    // Returns false otherwise.
}

bool trickster(string input) {
    string me = /* source code of trickster */;
    return !willAccept(me, input);
}
```
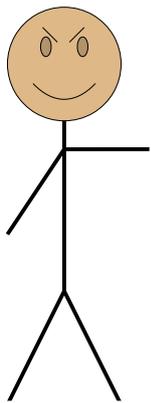
trickster
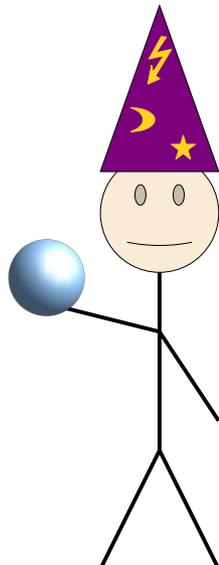
willAccept

trickster(input) returns true

↔

willAccept(me, input) returns true

↔

trickster(input) returns false

Theorem: $A_{TM} \notin \mathbf{R}$.

Proof: By contradiction; assume that $A_{TM} \in \mathbf{R}$. Then there is a decider D for A_{TM}. We can represent D as a function

```
bool willAccept(string function, string w);
```

that takes in the source code of a function `function` and a string `w`, then returns true if `function(w)` returns true and returns false otherwise. Given this, consider this function `trickster`:

```
bool trickster(string input) {
    string me = /* source code of trickster */;
    return !willAccept(me, input);
}
```

Since `willAccept` decides A_{TM} and `me` holds the source of `trickster`, we know that

  `willAccept(me, input)` returns true if and only if `trickster(input)` returns true.
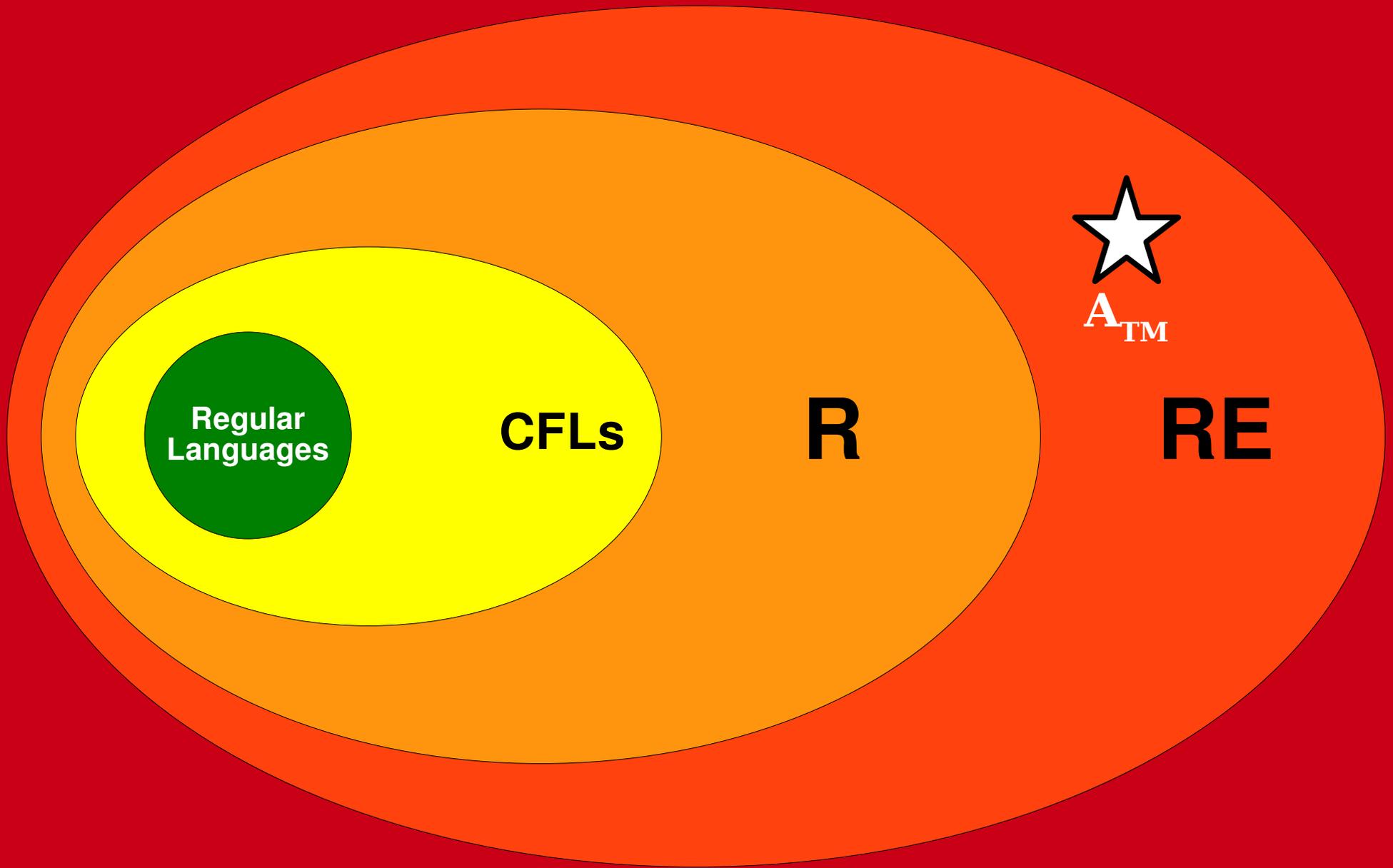
Given how `trickster` is written, we see that

  `willAccept(me, input)` returns true if and only if `trickster(input)` returns false.

This means that

    `trickster(input)` returns true if and only if `trickster(input)` returns false.

This is impossible. We've reached a contradiction, so our assumption was wrong and A_{TM} is undecidable. ∎

New Stuff!

More Impossibility Results

The Halting Problem

- The most famous undecidable problem is the ***halting problem***, which asks:

  **Given a TM *M* and a string *w*,
  will *M* halt when run on *w*?**

- As a formal language, this problem would be expressed as

  ***HALT* = { ⟨*M, w*⟩ | *M* is a TM that halts on *w* }**

- ***Theorem:*** *HALT* is recognizable, but undecidable.

  - There's a recognizer for *HALT*.
  - There is no decider for *HALT*.

HALT ∈ **RE**

- ***Claim:*** *HALT* ∈ **RE**.

- ***Idea:*** If you were certain that a TM *M* halted on a string *w,* could you convince me of that?

- Yes – just run *M* on *w* and see what happens!

```
bool willHalt(string TM, string w) {
    set up a simulation of M running on w;
    while (true) {
        if (M returned true) return true;
        else if (M returned false) return true;
        else simulate one more step of M running on w;
    }
}
```

***Theorem:*** The halting problem is undecidable.

A Decider for *HALT*

- Let's suppose that, somehow, we managed to build a decider for *HALT* = { ⟨*M, w*⟩ | *M* is a TM that halts on *w* }.

- Schematically, that decider would look like this:
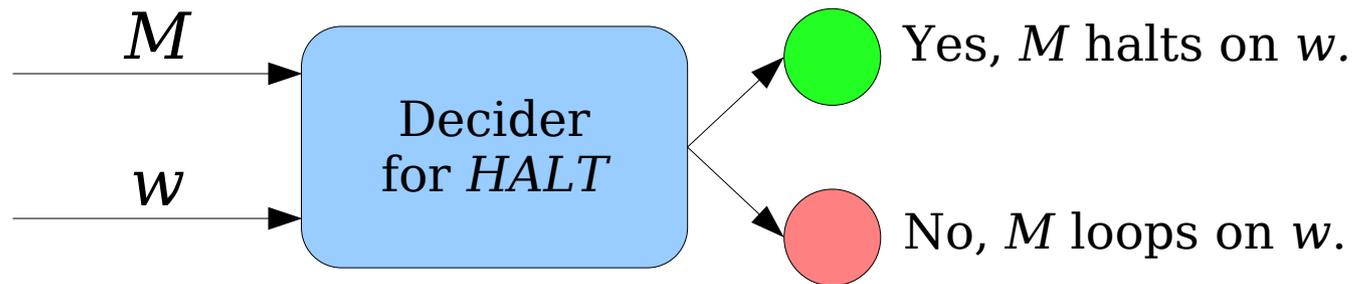
- We could represent this decider in software as a method

        bool willHalt(string function, string input);

  that takes as input a function `function` and a string `input`, then

  - returns true if `function(input)` returns anything (halts), and
  - returns false if `function(input)` never returns anything (loops).

```
bool willHalt(string function, string input) {
    // Returns true if function(input) halts.
    // Returns false otherwise.
}

bool trickster(string input) {
    string me = /* source code of trickster */;

    if (willHalt(me, input)) {
        while (true) {
            // Do nothing
        }
    } else {
        return true;
    }
}
```

trickster   willHalt
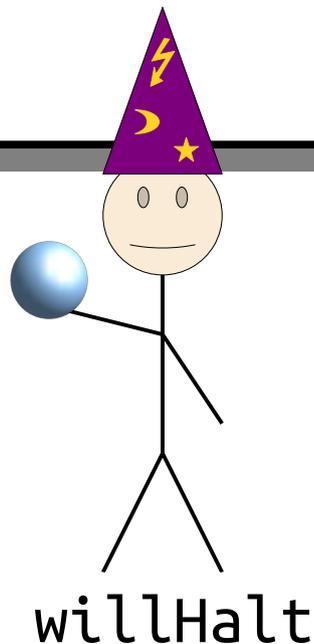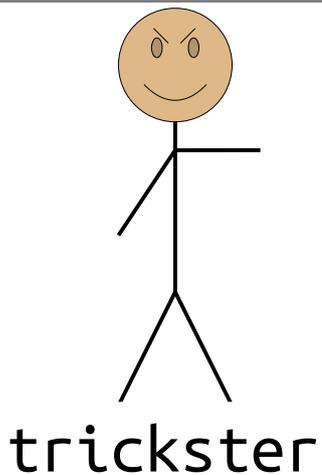
trickster(input) halts

↔

willHalt(me, input) returns true

↔

trickster(input) loops

Theorem: *HALT* ∉ **R**.

Proof: By contradiction; assume that *HALT* ∈ **R**. Then there is a decider *D* for *HALT*. We can represent *D* as a function

```
bool willHalt(string function, string w);
```

that takes in the source code of a function `function` and a string `w`, then returns true if `function(w)` halts and returns false otherwise. Given this, consider this function `trickster`:

```
bool trickster(string input) {
    string me = /* source code of trickster */;
    if (willHalt(me, input)) {
        while (true) { }
    } else {
        return true;
    }
}
```

Since `willHalt` decides *HALT* and `me` holds the source of `trickster`, we know that

> `willHalt(me, input)` returns true   if and only if   `trickster(input)` halts.
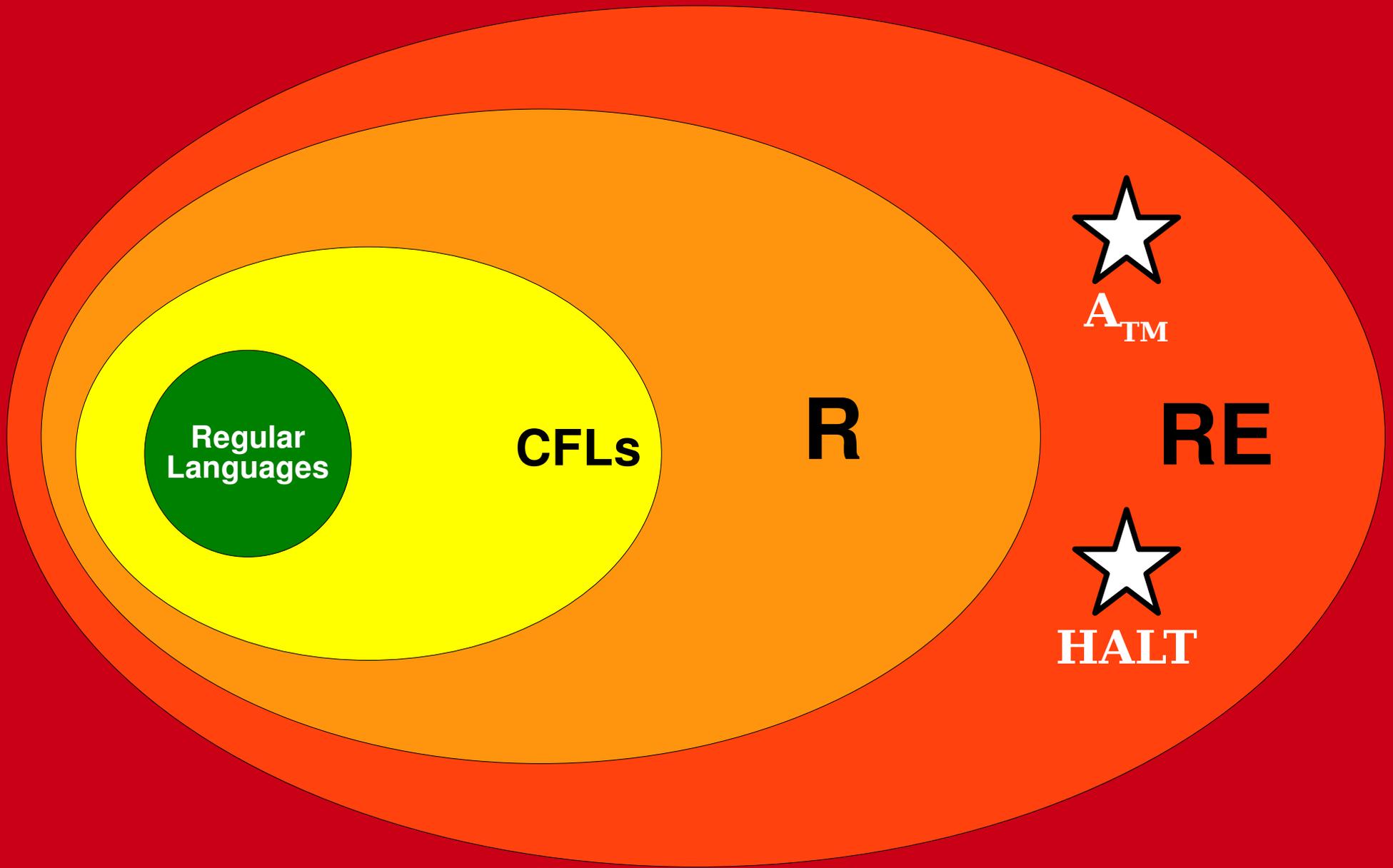
Given how `trickster` is written, we see that

> `willHalt(me, input)` returns true   if and only if   `trickster(input)` loops.

This means that

> `trickster(input)` halts   if and only if   `trickster(input)` loops.

This is impossible. We've reached a contradiction, so our assumption was wrong and *HALT* is undecidable. ∎
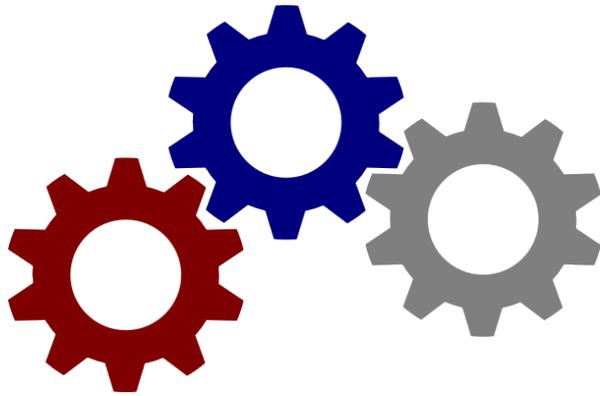
So What?

- These problems might not seem all that exciting, so who cares if we can't solve them?

- Turns out, this same line of reasoning can be used to show that some very important problems are impossible to solve.

Analogy Time!

Engineering Problem: Design a diesel engine that doesn't emit lots of NO$_x$ pollutants.

Engineering Prowess!
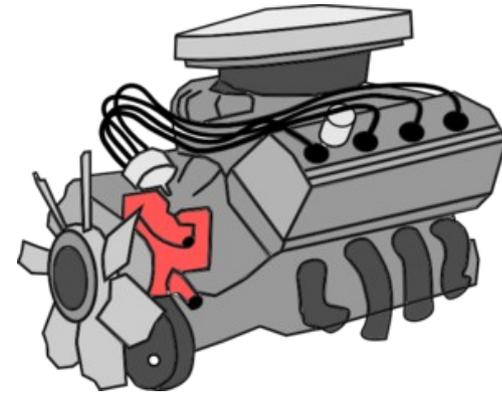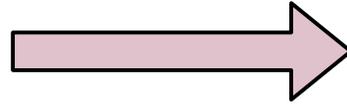
Awesome Engine!

Regulatory Problem: Design a testing procedure that, given a diesel engine, determines whether it emits lots of NO$_x$ pollutants.

Engine Testing Regimen

Yep

Nah
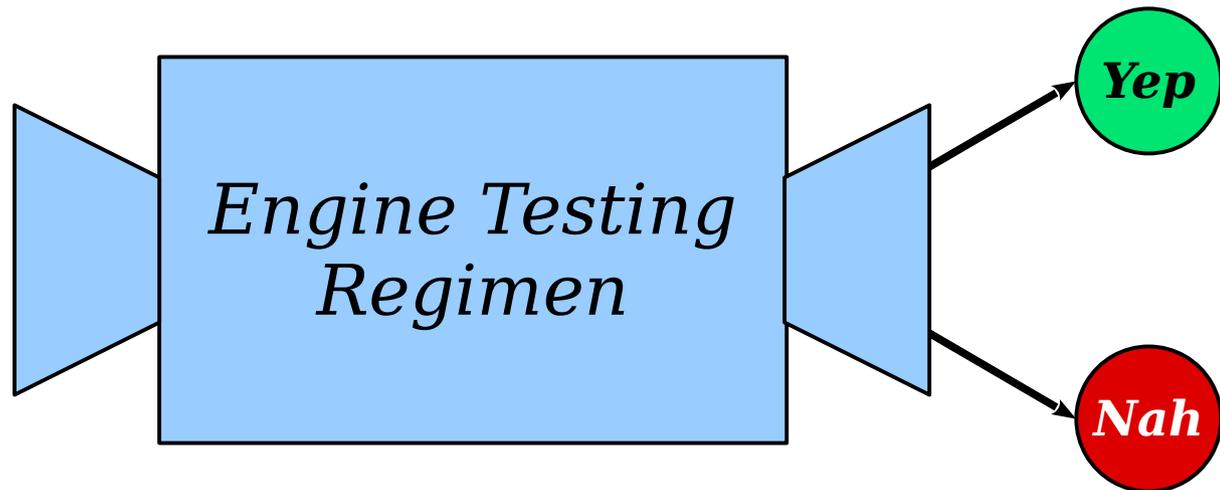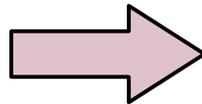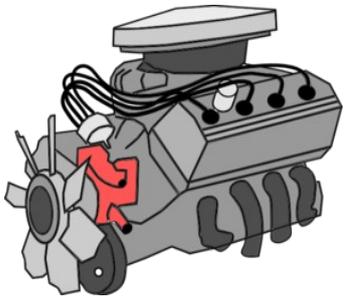
Fact: Almost all "regulatory problems" about computer programs are undecidable. That is, almost all problems of the form "does this program have [`behavioral property` X]" are undecidable.

This can be formalized through a result called **_Rice's Theorem_**; take CS154 for details!

A (Topical) Example

Secure Voting

- Suppose that you want to make a voting machine for use in an election between two parties (the ***Zomp Party*** and the ***Puce Party***).

- Let $\Sigma = \{$ z, p $\}$. A string $w \in \Sigma^*$ corresponds to a series of votes for the candidates.

- Example: zzpppzp means "two people voted for z, then three people voted for p, then one more person voted for z, then one more person voted for p."

Secure Voting

- A voting machine is a program that takes as input a string of z's and p's, then reports whether person z won the election.

- ***Question:*** Given a TM that someone claims is a secure voting machine, could we automatically check whether it actually is a secure voting machine?

A secure voting machine is a TM M where
M accepts $w \in \{z, p\}*$ if and only if w has more z's than p's.

```
bool bee(string input) {
    int numZs = countZsIn(input);
    int numPs = countPsIn(input);

    return numZs > numPs;
}
```

A (simple) secure voting machine.

```
bool topaz(string input) {
    return input != "" &&
           input[0] == 'z';
}
```

A (simple) insecure voting machine.

```
bool anna(string input) {
    int numZs = countZsIn(input);
    int numPs = countPsIn(input);

    if (numZs = numPs) {
        return false;
    } else if (numZs < numPs) {
        return false;
    } else {
        return true;
    }
}
```

An (evil) insecure voting machine.

```
bool green(string input) {
    int n = input.length();
    while (n > 1) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
    }

    int numZs = countZsIn(input);
    int numPs = countPsIn(input);

    return numZs > numPs;
}
```

No one knows!

Secure Voting

- A voting machine is a program that takes as input a string of z's and p's, then reports whether person z won the election.

- *Question:* Given a TM that someone claims is a secure voting machine, could we automatically check whether it actually is a secure voting machine?

A Decider for Secure Voting

- Let's suppose that, somehow, we managed to build a decider for the secure voting problem.

- Schematically, that decider would look like this:
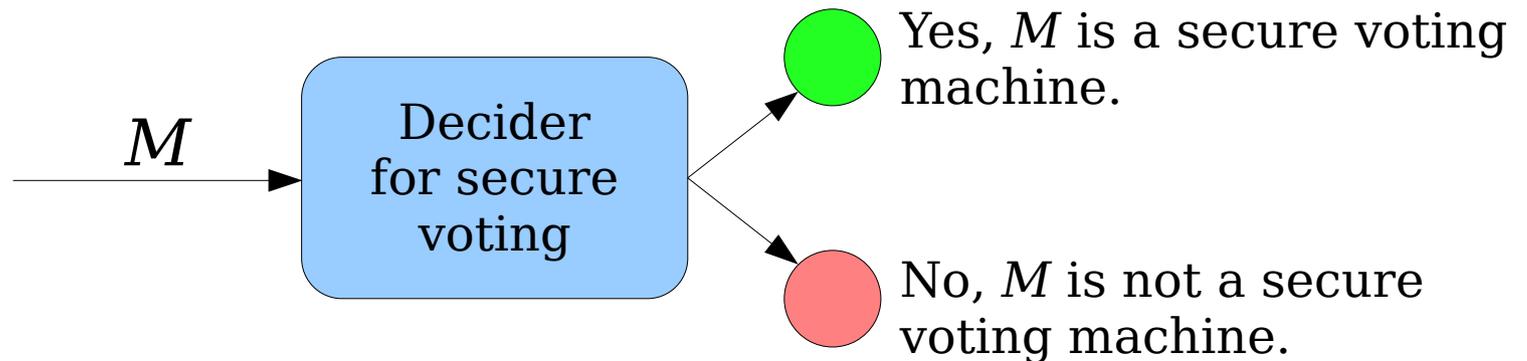
- We could represent this decider in software as a method

```
bool isSecureVotingMachine(string function);
```

that takes as input a function, then returns whether that function is a secure voting machine.

Theorem: The secure voting problem is undecidable.

Proof: By contradiction; there is a decider D for the secure voting problem. We can represent D as a function

```
bool isSecureVotingMachine(string function);
```

that takes in the source code of a function `function`, then returns whether function is a secure voting machine (that is, whether it accepts precisely the strings with more **z**'s than **p**'s). Given this, consider this function `trickster`:

```
bool trickster(string input) {
    string me = /* source code of trickster */;
    if (isSecureVotingMachine(me)) {
        return /* if input has at most as many z's as p's */;
    } else {
        return /* if input has more z's than p's */;
    }
}
```

Since `isSecureVotingMachine` decides the secure voting problem and `me` holds the source of `trickster`, we know that

  `isSecureVotingMachine(me)` returns true if and only if `trickster` is a secure voting machine.

Given how `trickster` is written, we see that

  `isSecureVotingMachine(me)` returns true if and only if `trickster` isn't a secure voting machine

This means that

  `trickster` is a secure voting machine if and only if `trickster` isn't a secure voting machine.

This is impossible. We've reached a contradiction, so our assumption was and the secure voting problem is undecidable. ∎

Interpreting this Result

- The previous argument tells us that *there is no general algorithm* that we can follow to determine whether a program is a secure voting machine. In other words, any general algorithm to check voting machines will always be wrong on at least one input.

- So what can we do?

    - Design algorithms that work in *some,* but not *all* cases. (This is often done in practice.)

    - Fall back on human verification of voting machines. (We do that too.)

    - Carry a healthy degree of skepticism about electronic voting machines. (Then again, did we even need the theoretical result for this?)

Time-Out for Announcements!

Problem Set Nine

- Problem Set Eight was due today at 2:30PM.

- Problem Set Nine goes out today. It's due next Friday at 2:30PM.

  - Play around with the limits of **R** and **RE** languages – the upper extent of computation!

  - See how everything fits together!

The Last Two Guides

- We've posted two final guides to the course website:

    - The ***Guide to Self-Reference***, which talks about proofs of undecidability via self-reference.

    - The ***Guide to the Lava Diagram***, which provides an intuition for how different classes of languages relate to one another.

- Give these a read – there's a ton of useful information in there!

Final Exam Logistics

- Our final exam is on ***Tuesday, December 13th*** from ***8:30AM – 11:30AM***. It'll be held in ***Hewlett 200***.

- The final exam is cumulative and covers topics from PS1 – PS9 and L00 – L27. The format is similar to that of the midterm, with a mix of short-answer questions and formal written proofs.

- Like the midterms, it's closed-book, closed-computer, and limited-note. You can bring one double-sided 8.5" × 11" notes sheet with you.

- Students with OAE accommodations: we will be reaching out later this evening to coordinate alternate final exam times.

Preparing for the Final Exam

- We've posted a gigantic compendium of CS103 practice problems on the course website.

- You can search for problems based on the topics they cover, whether solutions are available, whether they're ones we particularly like, whether they were used on past exams, and whether they have solutions.

- As always, *keep the TAs in the loop!* Ask us questions if you have them, feel free to stop by office hours to discuss solutions, etc.

Back to CS103!

Beyond **R** and **RE**

Beyond **R** and **RE**

- We've now seen how to use self-reference as a tool for showing undecidability (finding languages not in **R**).

- We still have not broken out of **RE** yet, though.

- To do so, we will need to build up a better intuition for the class **RE**.

What exactly is the class **RE**?

RE, Formally

- Recall that the class **RE** is the class of all recognizable languages:

    RE = { *L* | there is a TM *M* that recognizes *L* }

- Since **R** ≠ **RE**, there is no general way to "solve" problems in the class **RE**, if by "solve" you mean "make a computer program that can always tell you the correct answer."

- So what exactly *are* the sorts of languages in **RE**?

Example: Where's Waldo?
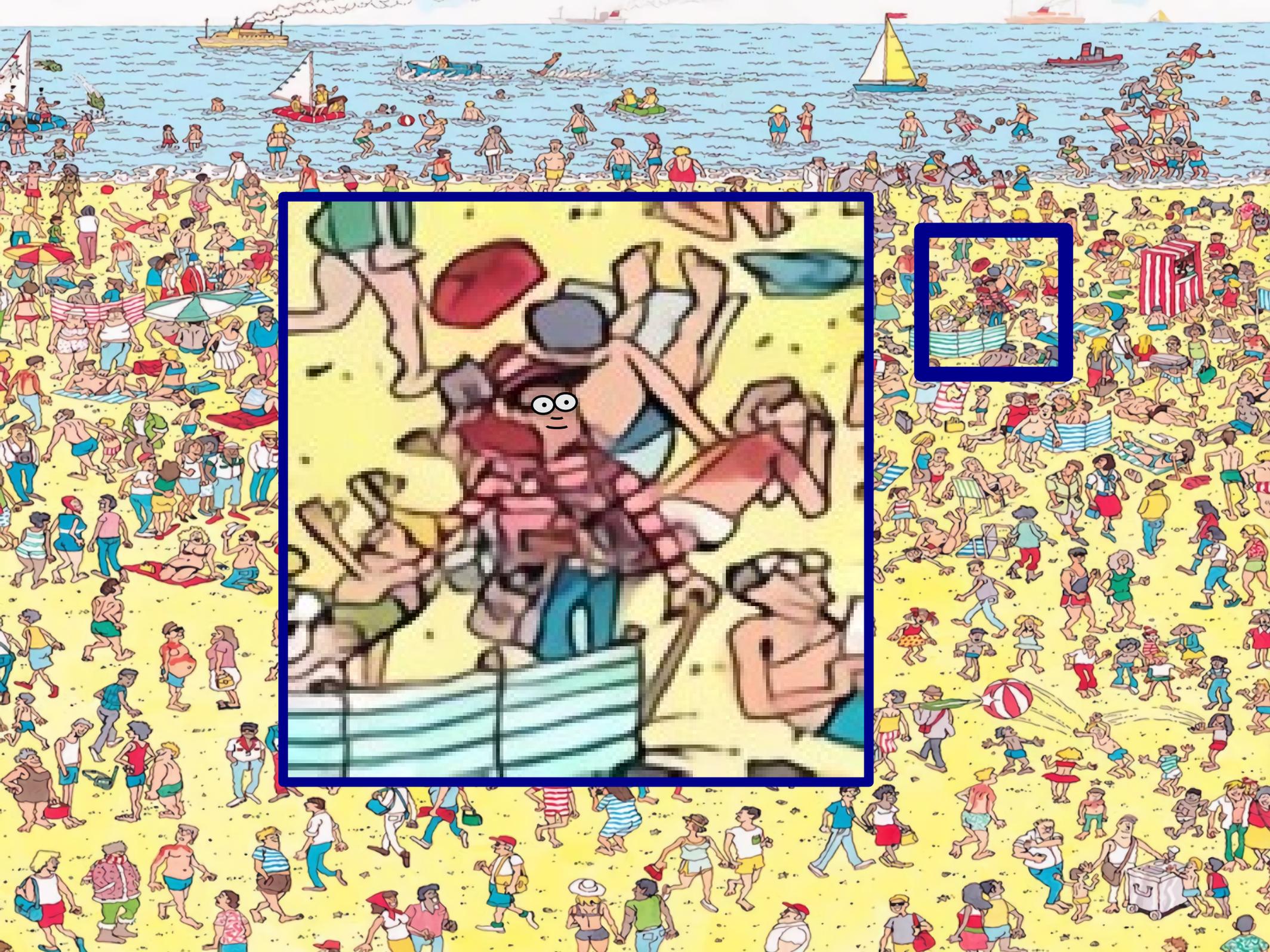
Key Intuition:

A language L is in **RE** if, for any string w, if you are *convinced* that $w \in L$, there is some way you could prove that to someone else.

Verification

2	5	7	9	6	4	1	8	3
4	9	1	8	7	3	6	5	2
3	8	6	1	2	5	9	4	7
6	4	5	7	3	2	8	1	9
7	1	9	5	4	8	3	2	6
8	3	2	6	1	9	5	7	4
1	6	3	2	5	7	4	9	8
5	7	8	4	9	6	2	3	1
9	2	4	3	8	1	7	6	5

Does this Sudoku puzzle
have a solution?

Verification

11

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

1

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

But also...
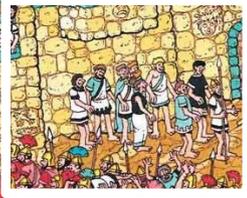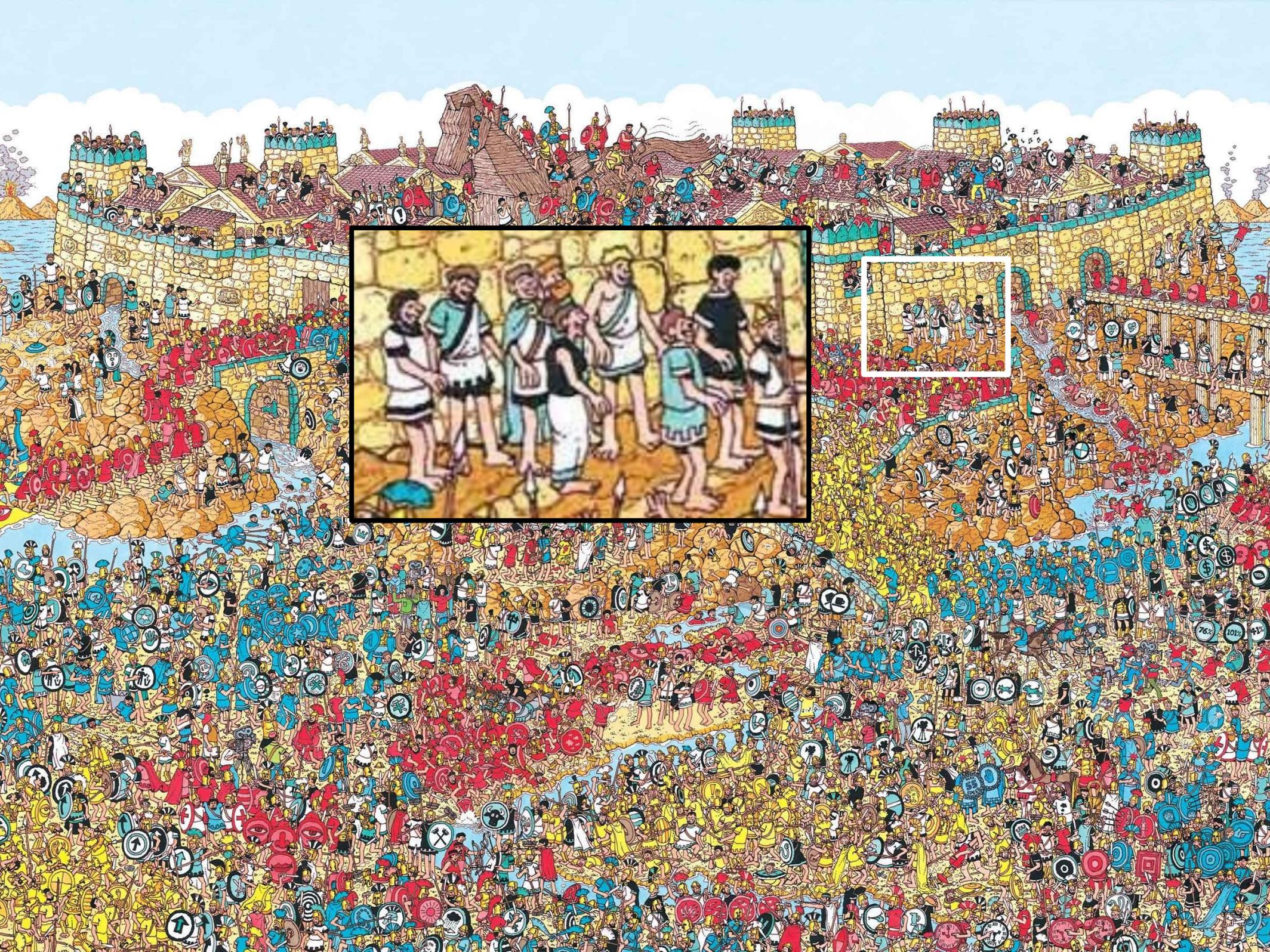
Verification

1	1	7	1	6	1	1	1	1
1	1	1	1	1	3	1	5	2
3	1	1	1	1	5	9	1	7
6	1	5	1	3	1	8	1	9
1	1	1	1	1	1	1	2	1
8	1	2	1	1	1	5	1	4
1	1	3	2	1	7	1	1	8
5	7	1	4	1	1	1	1	1
1	1	4	1	8	1	7	1	1

Does this Sudoku puzzle
have a solution?

Verification

11

Does the hailstone sequence
terminate for this number?

Verifiers

- A ***verifier*** for a language *L* is a TM *V* with the following two properties:

  V halts on all inputs.

  ∀w ∈ Σ*. (w ∈ L ↔ ∃c ∈ Σ*. V accepts ⟨w, c⟩)

- Intuitively, what does this mean?

Deciders and Verifiers

"Solve the problem"

input string (w) → Decider M for L

yes!

no!

If M accepts, then $w \in L$.

If M rejects, then $w \notin L$.

M halts on all inputs.
$w \in L \leftrightarrow M$ accepts w

"Check an answer"

input string (w) →
certificate (c) → Verifier V for L

yes!

not sure

If V accepts $\langle w, c \rangle$, then $w \in L$.

If V rejects $\langle w, c \rangle$, **we don't know** whether $w \in L$.

V halts on all inputs.
$w \in L \leftrightarrow \exists c \in \Sigma^*. V$ accepts $\langle w, c \rangle$

Verifiers

- A ***verifier*** for a language *L* is a TM *V* with the following properties:

V halts on all inputs.

∀*w* ∈ Σ*. (*w* ∈ L   ↔   ∃*c* ∈ Σ*. V accepts ⟨*w, c*⟩)

- Some notes about *V*:

  - If *V* accepts ⟨*w, c*⟩, we're guaranteed *w* ∈ *L*.

  - If *V* rejects ⟨*w, c*⟩, then either

    – *w* ∈ *L*, but you gave the wrong *c*, or
    – *w* ∉ *L*, so no possible *c* will work.
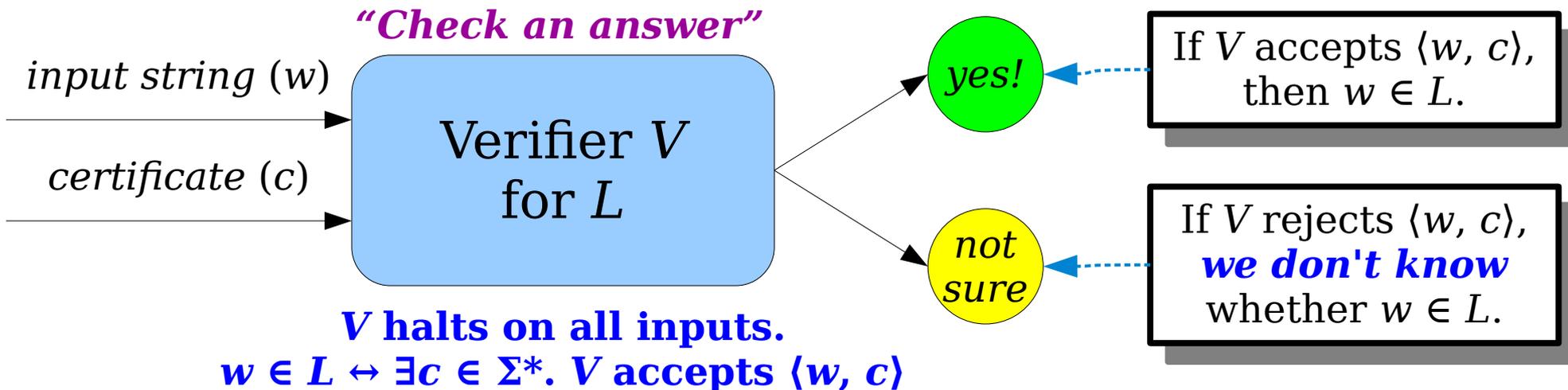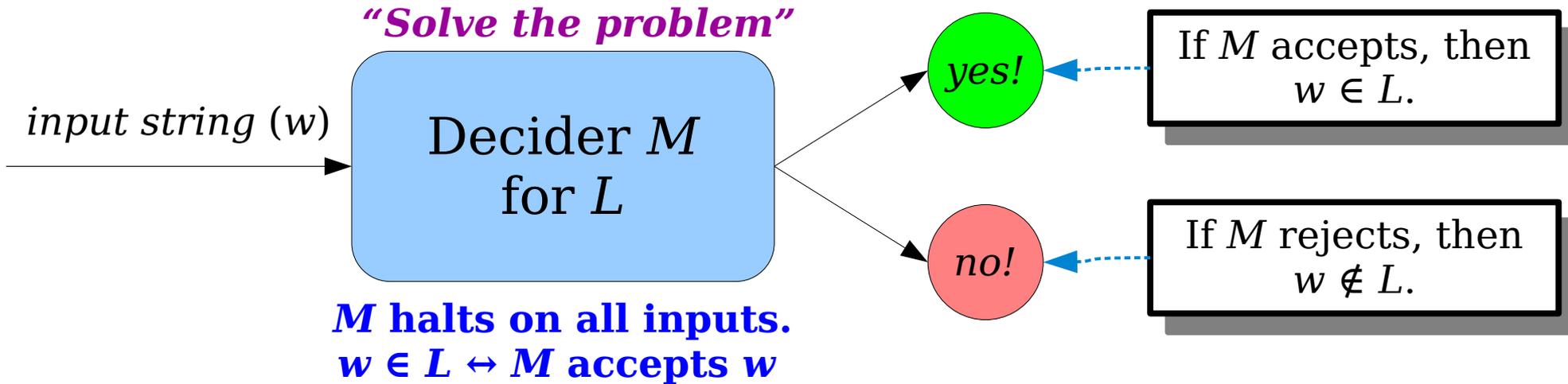
Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:

$$V \textbf{ halts on all inputs.}$$

$$\forall w \in \Sigma^*. \ (w \in L \ \leftrightarrow \ \exists c \in \Sigma^*. \ V \textbf{ accepts } \langle w, c \rangle)$$

- Some notes about V:

  - Notice that the certificate c is existentially quantified. Any string $w \in L$ must have at least one c that causes V to accept, and possibly more.

  - V is required to halt, so given any potential certificate c for w, you can check whether the certificate is correct.

Verifiers

- A ***verifier*** for a language *L* is a TM *V* with the following properties:

_V_ halts on all inputs.

∀*w* ∈ Σ*. (*w* ∈ *L* ↔ ∃*c* ∈ Σ*. *V* accepts ⟨*w, c*⟩)

- Some notes about *V*:

  - Notice that *V* isn't a decider for *L* and isn't a recognizer for *L*.

  - The job of *V* is just to check certificates, not to decide membership in *L*.

Verifiers

- A ***verifier*** for a language *L* is a TM *V* with the following properties:

V halts on all inputs.

∀*w* ∈ Σ*. (*w* ∈ *L* ↔ ∃*c* ∈ Σ*. *V* accepts ⟨*w*, *c*⟩)

- Some notes about *V*:

  - Although this formal definition works with a string *c*, remember that *c* can be an encoding of some other object.

  - In practice, *c* will likely just be "some other auxiliary data that helps you out."

A Sample Verifier

- Let L be the following language:

  $L = \{\ \langle n \rangle \mid n \in \mathbb{Z} \text{ and } n \text{ is the sum of three cubes } \}$

- Let's see how to build a verifier for L. Our verifier will take as input

  - a natural number n, and

  - a certificate c.

- The certificate c should be some evidence that suggests that n is the sum of three cubes.

- What information could we put into the certificate?

A Sample Verifier

- Let L be the following language:

  $L = \{ \langle n \rangle \mid n \in \mathbb{Z}$ and n is the sum of three cubes $\}$

- Here's a verifier for L:

```
bool checkCubeSum(int n, int x, int y, int z) {
    return x*x*x + y*y*y + z*z*z == n;
}
```
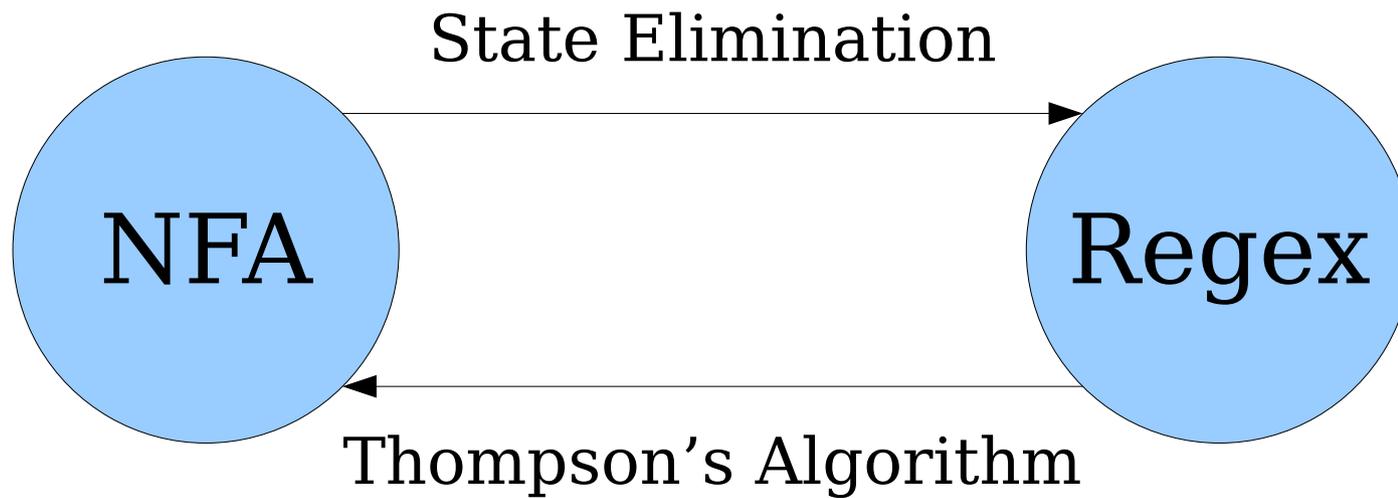
- Do you see why $\langle n \rangle \in L$ if and only if are x, y, z where checkCubeSum(G, c) returns true?

- Do you see why checkCubeSum always halts?

What languages are verifiable?

Theorem: If L is a language, then there is a verifier for L if and only if $L \in$ **RE**.

Where We've Been

Where We're Going
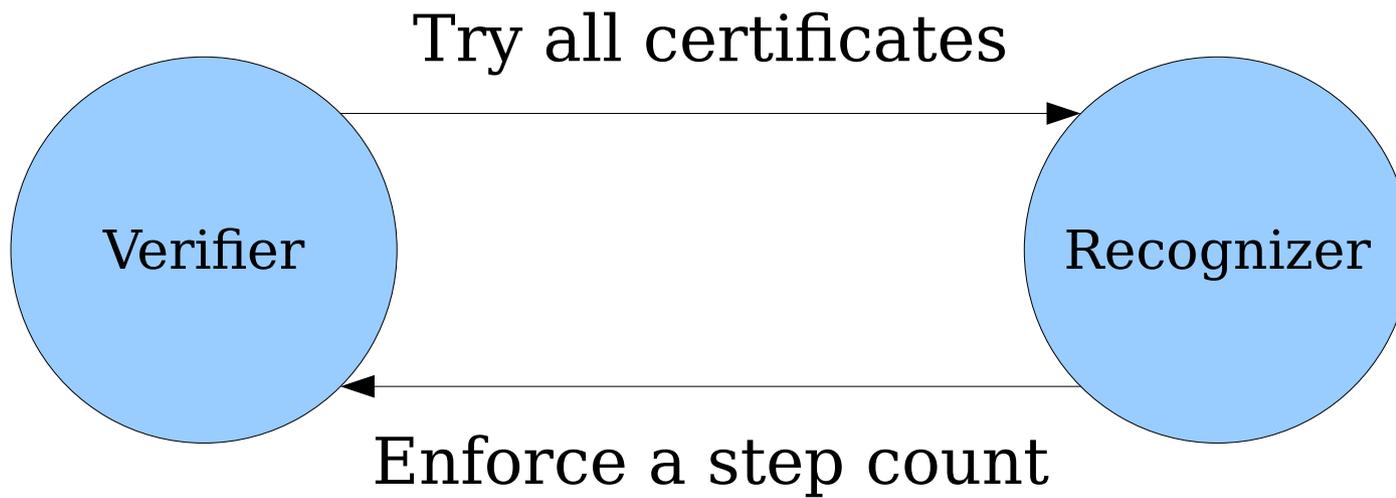
Verifiers and **RE**

- ***Theorem:*** If there is a verifier V for a language L, then $L \in \textbf{RE}$.

- ***Proof goal:*** Given a verifier V for a language L, find a way to construct a recognizer M for L.

"Check an answer"

input string (w) → Verifier V for L → *yes!*

certificate (c) → Verifier V for L → *not sure*

Verifiers and **RE**

- ***Theorem:*** If there is a verifier V for a language L, then $L \in$ **RE**.

- ***Proof goal:*** Given a verifier V for a language L, find a way to construct a recognizer M for L.
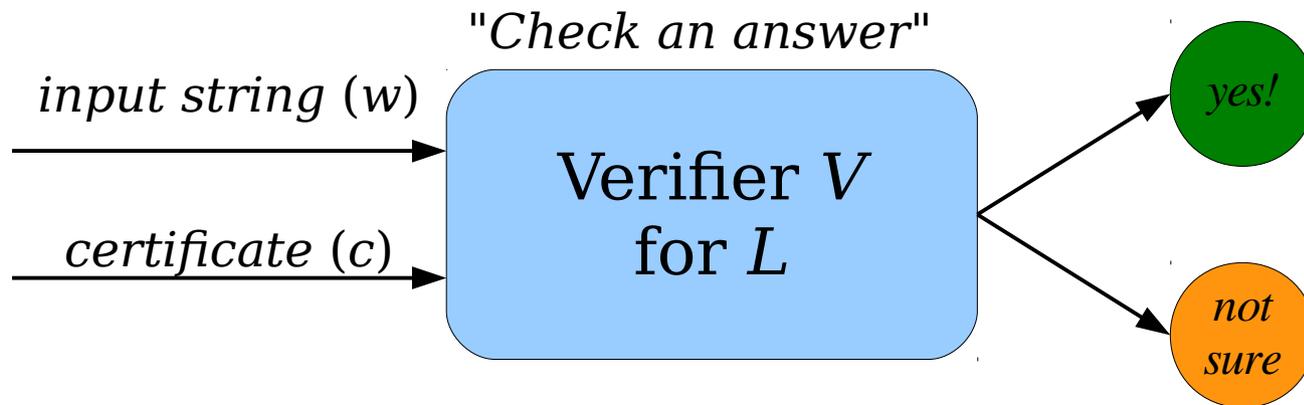
Verifiers and **RE**

- ***Theorem:*** If V is a verifier for L, then $L \in$ **RE**.

- ***Proof sketch:*** Consider the following program:

```
bool isInL(string w) {
    for (each string c) {
        if (V accepts ⟨w, c⟩) return true;
    }
}
```
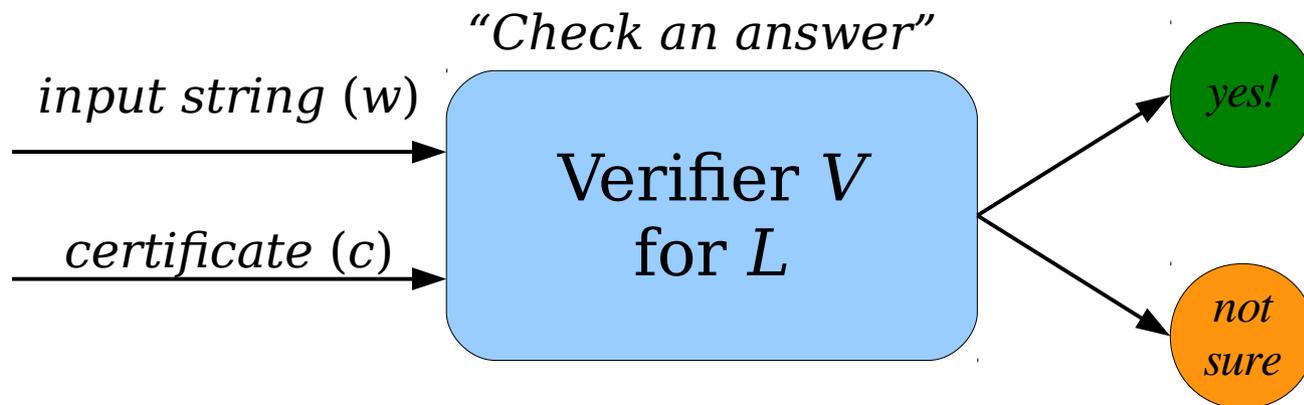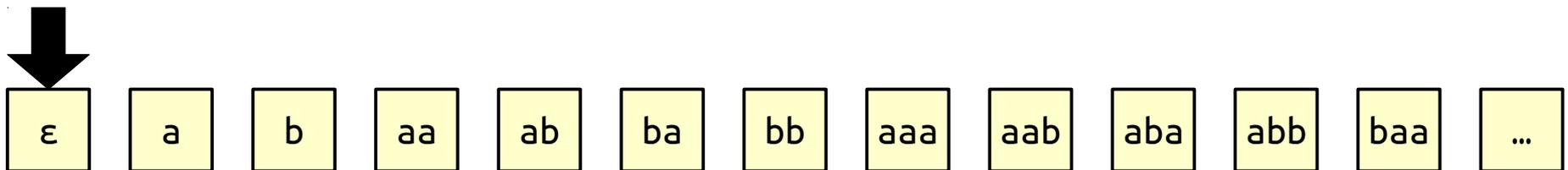
If $w \in L$, there is some $c \in \Sigma^*$ where V accepts $\langle w, c \rangle$. The function `isInL` tries all possible strings as certificates, so it will eventually find c (or some other working certificate), see V accept $\langle w, c \rangle$, then return true. Conversely, if `isInL(w)` returns true, then there was some string c such that V accepted $\langle w, c \rangle$, so we see that $w \in L$. ∎

Verifiers and **RE**

- ***Theorem:*** If $L \in \textbf{RE}$, then there is a verifier for L.

- ***Proof Goal:*** Beginning with a recognizer M for the language L, show how to construct a verifier V for L.

Verifiers and **RE**

- **_Theorem:_** If $L \in$ **RE**, then there is a verifier for L.

- **_Proof sketch:_** Let L be a **RE** language and let M be a recognizer for it. Consider this function:

```
bool checkIsInL(string w, int c) {
    TM M = /* hardcoded version of a recognizer for L */;
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
        simulate the next step of M running on W;
    }
    return whether M is in an accepting state;
}
```

Note that checkIsInL always halts, since each step takes only finite time to complete. Next, notice that if there is a c where checkIsInL(w, c) returns true, then M accepted w after running for c steps, so $w \in L$. Conversely, if $w \in L$, then M accepts w after some number of steps (call that number c). Then checkIsInL(w, c) will run M on w for c steps, watch M accept w, then return true. ■

RE and Proofs

- Verifiers and recognizers give two different perspectives on the "proof" intuition for **RE**.

- Verifiers are explicitly built to check proofs that strings are in the language.

  - If you know that some string w belongs to the language and you have the proof of it, you can convince someone else that $w \in L$.

- You can think of a recognizer as a device that "searches" for a proof that $w \in L$.

  - If it finds it, great!

  - If not, it might loop forever.

RE and Proofs

- If the **RE** languages represent languages where membership can be proven, what does a non-**RE** language look like?

- Intuitively, a language is *not* in **RE** if there is no general way to prove that a given string $w \in L$ actually belongs to L.

- In other words, even if you knew that a string was in the language, you may never be able to convince anyone of it!

Finding Non-**RE** Languages

Finding Non-**RE** Languages

- Right now, we know that non-**RE** languages exist, but we have no idea what they look like.

- How might we find one?

Recognizers and Recognizability

- ***Recall:*** We say that *M* is a recognizer for *L* if the following is true:

  ∀*w* ∈ Σ*. (*w* ∈ *L*   ↔   *M* accepts *w*).

- This above description applies to all strings, including strings that, by pure coincidence, happen to be the source code of a TM.

- What happens if we list off all Turing machines, looking at how those TMs behave given other TMs as input?

M_0

M_1

M_2

M_3

M_4

M_5

...
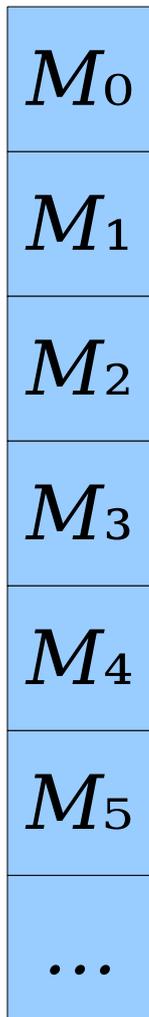
All Turing machines, listed in some order.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...	...	...	...	...	...	...	...

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...	...	...	...	...	...	...	...

Acc	Acc	Acc	No	Acc	No	...

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...	...	...	...	...	...	...	...

No	No	No	Acc	No	Acc	...

No TM has this behavior!

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...	...	...	...	...	...	...	...

No	No	No	Acc	No	Acc	...

{ $\langle M \rangle \mid M$ is a TM that does not accept $\langle M \rangle$ }

Diagonalization Revisited

- The ***diagonalization language***, which we denote L_D, is defined as

  $$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } M \text{ does not accept } \langle M \rangle \}$$

- We constructed this language to be different from the language of every TM.

- Therefore, $L_D \notin \mathbf{RE}$! Let's go prove this.

L_D = { $\langle M \rangle$ | M is a TM and M does not accept $\langle M \rangle$ }

Theorem: $L_D \notin \mathbf{RE}$.

Proof: Assume for the sake of contradiction that $L_D \in \mathbf{RE}$. This means that there is a recognizer R for L_D.

Now, focus on what happens if we run recognizer R on its own encoding (that is, running R on $\langle R \rangle$). Since R is a recognizer for L_D, we see that

$$R \text{ accepts } \langle R \rangle \qquad \text{if and only if} \qquad \langle R \rangle \in L_D.$$
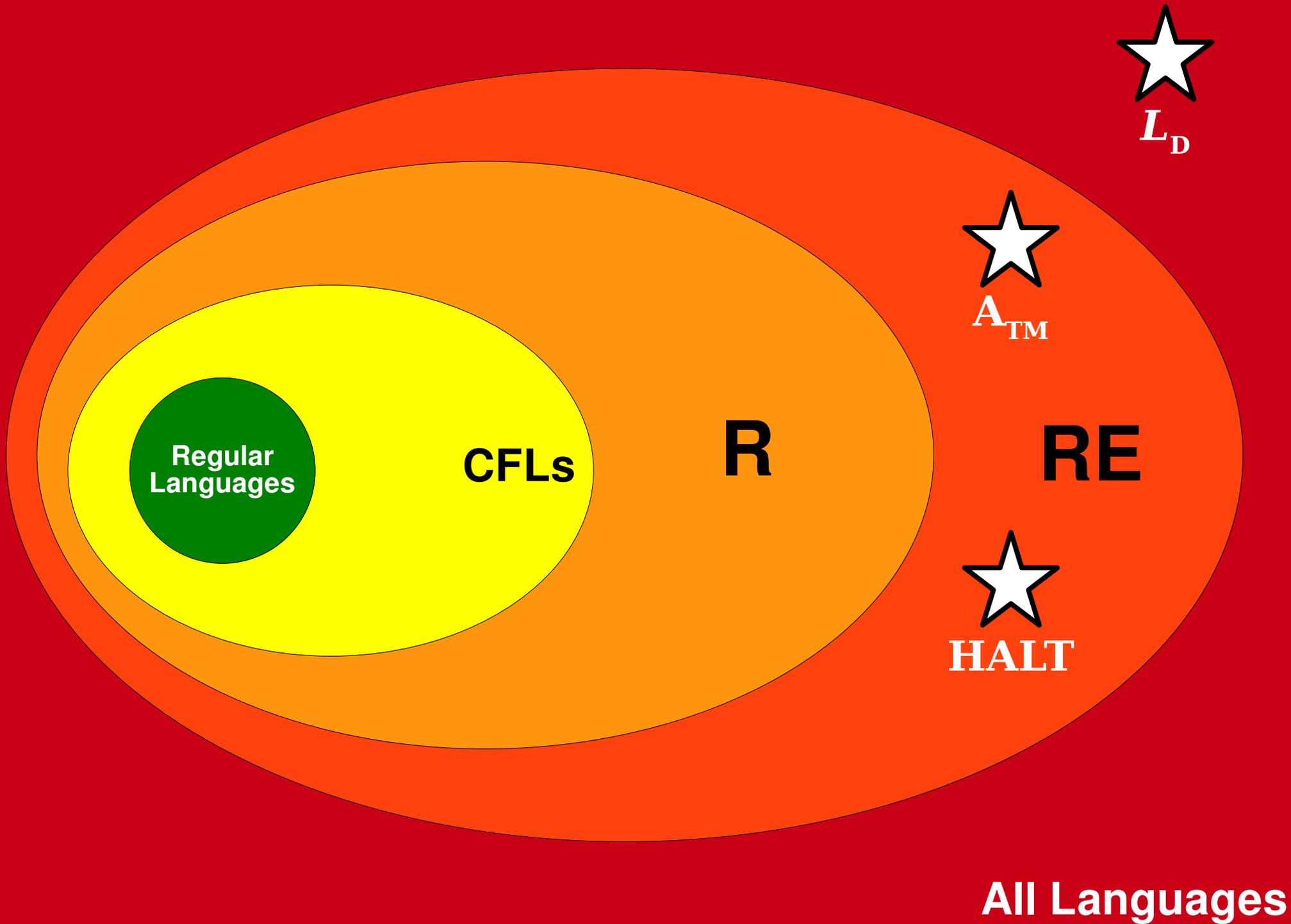
By definition of L_D, we know that

$$\langle R \rangle \in L_D \qquad \text{if and only if} \qquad R \text{ does not accept } \langle R \rangle.$$

Combining the two above statements tells us that

$$R \text{ accepts } \langle R \rangle \qquad \text{if and only if} \qquad R \text{ does not accept } \langle R \rangle.$$

This is impossible. We've reached a contradiction, so our assumption was wrong, and so $L_D \notin \mathbf{RE}$. ∎

What This Means

- On a deeper philosophical level, the fact that non-**RE** languages exist supports the following claim:

    ***There are statements that
    are true but not provable.***

- Intuitively, given any non-**RE** language, there will be some string in the language that *cannot* be proven to be in the language.

- This result can be formalized as a result called ***Gödel's incompleteness theorem***, one of the most important mathematical results of all time.

- Want to learn more? Take Phil 152 or CS154!

What This Means

- On a more philosophical note, you could interpret the previous result in the following way:

  There are inherent limits about what mathematics can teach us.

- There's no automatic way to do math. There are true statements that we can't prove.

- That doesn't mean that mathematics is worthless. It just means that we need to temper our expectations about it.

Where We Stand

- We've just done a crazy, whirlwind tour of computability theory:
  - *The Church-Turing thesis* tells us that TMs give us a mechanism for studying computation in the abstract.
  - *Universal computers* – computers as we know them – are not just a stroke of luck. The existence of the universal TM ensures that such computers must exist.
  - *Self-reference* is an inherent consequence of computational power.
  - *Undecidable problems* exist partially as a consequence of the above and indicate that there are statements whose truth can't be determined by computational processes.
  - *Unrecognizable problems* are out there and can be discovered via diagonalization. They imply there are limits to mathematical proof.

The Big Picture
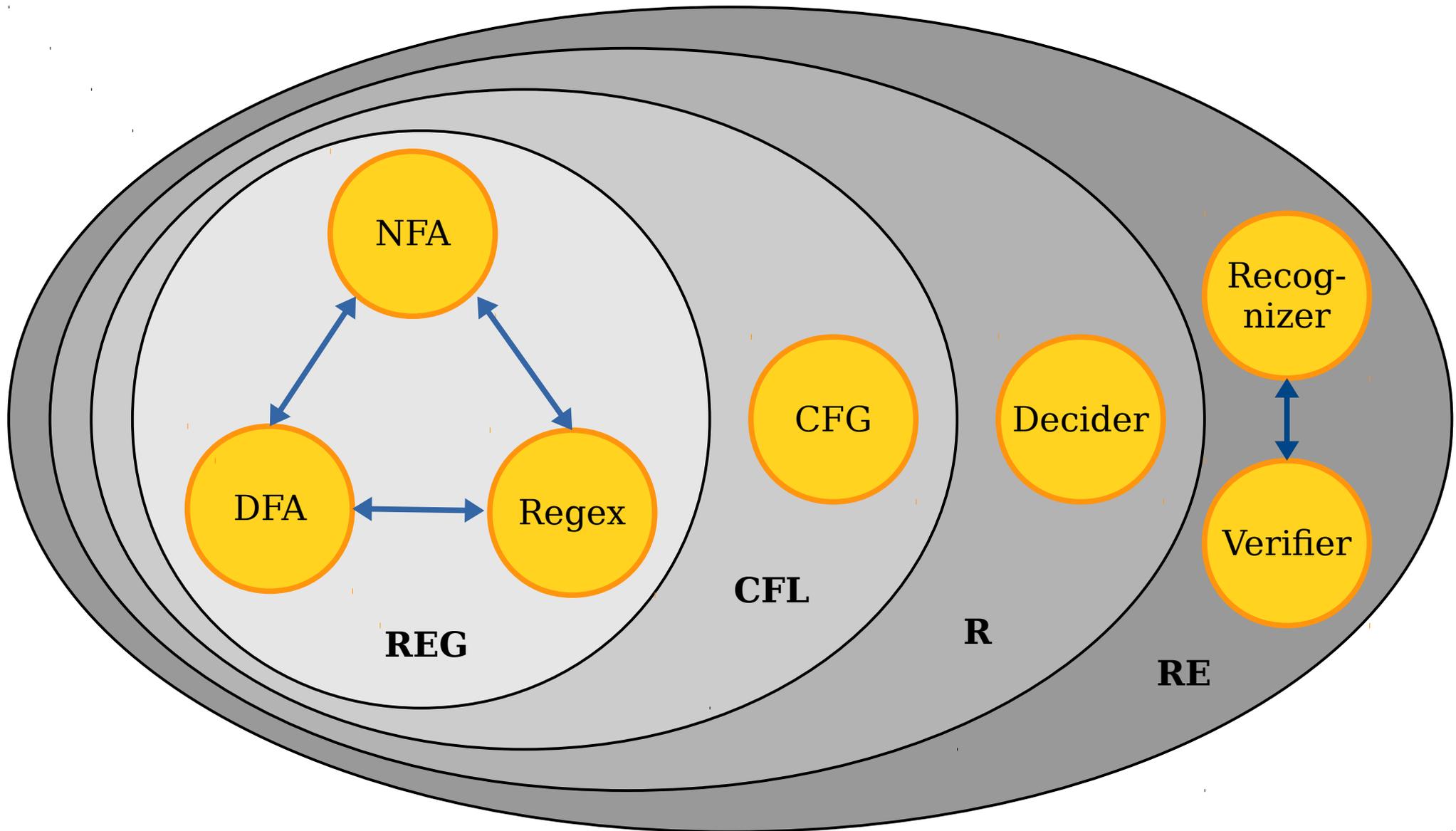
Where We've Been

- The class **R** represents problems that can be solved by a computer.

- The class **RE** represents problems where "yes" answers can be verified by a computer.

Where We're Going

- The class **P** represents problems that can be solved *efficiently* by a computer.

- The class **NP** represents problems where "yes" answers can be verified *efficiently* by a computer.

Next Time

- ***Introduction to Complexity Theory***

  - Not all decidable problems are created equal!

- ***The Classes P and NP***

  - Two fundamental and important complexity classes.

- ***The P $\stackrel{?}{=}$ NP Question***

  - A literal million-dollar question!